Quiz 2

COL 352
Introduction to Automata & Theory of Computation

Problem 1

Given the grammar

$$S \to AA$$

$$A \to AAA \mid a \mid bA \mid Ab$$

is it true that it generates all strings with even number of a's? Either give a counter-example or state a formal induction assertion for S and A (induction proof not required) that makes the claim true.

Solution : Yes, all the strings with even number of a's can be generated

Induction Assertion : $\forall i \geq 1$ and $\forall w, |w| = i$

1. $A \to w$ iff w contains odd number of a's
2. $S \to w$ iff w contains even number of a's

I) Claim : $\forall i \geq 1$ and $\forall w, |w| = i$, $A \to w$ iff w contains odd number of a's

Proof : By induction on the length of string w

Basis : $A \to a$. Since A terminates only at a, a is the only 1 length string that can be generated

Induction Hypothesis : The grammar generates all strings of length $n \leq k$ iff number of a's is odd

Induction Step: For $|w| = k+1$ and has odd number of a's then for $A \to w$ - w can be of the form aw_1a, bw_1b, aw_2b or bw_2a, where

(i) $|w_1| = k-2$ and has odd $a's$

$$A \to AAA \to aAA \to aA_1a$$

$$A \to bA \to bA_1b$$

From Induction Hypothesis, $A_1 \xrightarrow{*} w_1$

(ii) $|w_2| = k-2$ and has even number of a's

$$A \to AAA \to AAAb \to aA_1A_2b$$

$$A \to AAA \to bAAA \to bA_1A_2a$$

Let w_{21} and w_{22} be such that $|w_{21}| < k-2$, $|w_{22}| < k-2$ and $|w_{21}| + |w_{22}| = k-2$ and odd a's.

From IH, $A_1 \xrightarrow{*} w_{21}$ and $A_2 \xrightarrow{*} w_{22}$, i.e., $w_{21}w_{22} = w_2$ and has even a's.

Thus $A_1A_2 \xrightarrow{*} w_2$

If $A \to w$ and $|w| = k+1$, then w has odd number of a's - Let w be a $k+1$ length string with even a's.

Possible grammar productions for generating $n = k+1$ length string - $A \to bA_1$ and $A \to A_1b$

A_1 should generate string of length k with even a's. But from IH, we have that only those strings of length k are generated by A which have odd a's. Thus contradiction.

$A \rightarrow A_2 A_3 A_4$

Also, from IH, A_2, A_3 and A_4 generates strings of length less than k with odd number of a's.

Thus, together they generate a string with odd number of a's as 3 odds will make 1 odd.

Thus for all $i \geq 1$, and all strings $|w| = i$, $A \rightarrow w$ iff w contains odd number of a's

II) Claim : $\forall i \geq 1$ and $\forall w, |w| = i$, $S \rightarrow w$ iff w contains even number of a's

Proof : Any string, x with even a's (say n) can be divided into two strings with odd a's.

Since A generates all strings with odd a's, $S \rightarrow AA$ can generate all string with even number of a's

From I and II, we can prove that the grammar generates all strings with even number of a's

Problem 2

Let L be the language of all strings over the alphabet $\{a, b, c, d\}$ that do not contain an equal number of a, b, c, d's, i.e. $aabcc \in L$, but $abccadbd \notin L$. Is L a CFL? Justify your answer

Solution : Let $num(x, w)$ denote the number of x's in the word w, $x \in \Sigma$ $(= \{a, b, c, d\})$

Consider the language $L_{xy} = \{w \mid num(x, w) \neq num(y, w)\}$

Clearly, $L = \bigcup_{x \neq y}^{x,y \in \Sigma} L_{xy}$

Further, $\forall x, y \in \Sigma,\ L_{xy}$ is a CFL, with the CFG

$$S \rightarrow U \mid V$$

$$U \rightarrow TxU \mid TxT$$

$$V \rightarrow TyV \mid TyT$$

$$T \rightarrow xTyT \mid yTxT \mid tT \mid \epsilon$$

where $t \in \Sigma - \{x, y\}$ $(= \{c, d\}$ for $x = a$ and $y = b)$. Here, the symbol U generates strings w with $num(x, w) > num(y, w)$. V generates $num(y, w) > num(x, w)$ and T generates $num(x, w) = num(y, w)$.

Alternately, L_{xy} can be represented by the PDA as follows -

- If stack is empty or top has x on it, push x onto the stack when it is read, pop from stack when y is
- If stack is empty or top has y on it, push y onto the stack when it is read, pop from stack when x is
- Simply consume symbols other than x and y
- Transition to the final, accepting state if the whole string has been consumed but the stack is non-empty

Since the finite union of CFLs yields a CFL (simply another transition from the start symbol), L is a CFL

Notes on grading -

Correctly identifying CFL is worth 2/10 marks

$L' = \{w \mid num(a, w) \neq num(b, w) \neq num(c, w) \neq num(d, w)\} \subset L$, 0 if $L' = L$,